

MEASURES OF IRRATIONALITY FOR HYPERSURFACES OF LARGE DEGREE

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Dedicated to János Kollár on the upcoming occasion of his sixtieth birthday

INTRODUCTION

The purpose of this paper is to study various measures of irrationality for hypersurfaces of large degree in projective space and other varieties. The theme is that positivity properties of canonical bundles lead to lower bounds for these invariants. In particular, we prove the conjecture of [4] that if $X \subseteq \mathbf{P}^{n+1}$ is a very general smooth hypersurface of dimension n and degree $d \geq 3n$, then any dominant rational mapping $f : X \dashrightarrow \mathbf{P}^n$ must satisfy

$$\deg(f) \geq d - 1,$$

with equality if and only if f is given by projection from a point of X .¹

To start with some background, recall that the *gonality* $\text{gon}(C)$ of an irreducible complex projective curve C is defined to be the least degree of a branched covering

$$C' \longrightarrow \mathbf{P}^1,$$

where C' is the normalization of C . Thus

$$\text{gon}(C) = 1 \iff C \approx_{\text{birat}} \mathbf{P}^1,$$

and it is profitable in general to view $\text{gon}(C)$ as measuring the failure of C to be rational. Because of this there has been a certain amount of interest over the years in bounding from below the gonality of various natural classes of curves. For instance, a classical theorem of Noether states that if $C \subseteq \mathbf{P}^2$ is a smooth plane curve of degree $d \geq 3$, then

$$\text{gon}(C) = d - 1,$$

with the relevant coverings given by projection from a point of C . This was generalized to complete intersection and other curves in [14, Ex. 4.12] and [10] by means of vector bundle techniques. In another direction, Abramovich [1] used results of Li and Yau to obtain a linear lower bound on the gonality of modular curves, and the behavior of gonality in certain towers of coverings was studied by Hwang and To [12] as a consequence of relations they

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¹It was actually conjectured in [4] that this statement holds as soon as $d \geq 2n + 1$. In their Appendix [5] to the present paper, Bastianelli and De Poi show that our hypothesis $d \geq 3n$ can be relaxed to $d \geq 3n - 2$.

established between gonality and injectivity radii; the paper [3] contains some interesting applications of these results.

Several authors have proposed and studied some analogous measures of irrationality for an irreducible complex projective variety X of arbitrary dimension n . We will be concerned here with three of these – the *degree of irrationality*, the *connecting gonality*, and the *covering gonality* of X – defined as follows:

$$\text{irr}(X) = \min \left\{ \delta > 0 \mid \begin{array}{c} \exists \text{ degree } \delta \text{ rational covering} \\ X \dashrightarrow \mathbf{P}^n \end{array} \right\};$$

$$\text{conn. gon}(X) = \min \left\{ c > 0 \mid \begin{array}{c} \text{General points } x, y \in X \text{ can be} \\ \text{connected by an irreducible curve} \\ C \subseteq X \text{ with } \text{gon}(C) = c. \end{array} \right\};$$

$$\text{cov. gon}(X) = \min \left\{ c > 0 \mid \begin{array}{c} \text{Given a general point } x \in X, \exists \text{ an} \\ \text{irreducible curve } C \subseteq X \text{ through } x \\ \text{with } \text{gon}(C) = c. \end{array} \right\}.$$

(Note that the curves C computing the connecting and covering gonalitys are allowed to be singular.) Thus

$$\begin{aligned} \text{irr}(X) &= 1 \iff X \text{ is rational,} \\ \text{conn. gon}(X) &= 1 \iff X \text{ is rationally connected,} \\ \text{cov. gon}(X) &= 1 \iff X \text{ is uniruled,} \end{aligned}$$

and in general one has the inequalities

$$(1) \quad \text{cov. gon}(X) \leq \text{conn. gon}(X) \leq \text{irr}(X).$$

The integer $\text{irr}(X)$ is perhaps the most natural generalization of the gonality of a curve, but $\text{cov. gon}(X)$ often seems to be easier to control.²

The degree of irrationality was introduced by Heinzer and Moh in [11], and Yoshihara subsequently computed it for several classes of surfaces ([22], [18], [23], [24]). Lopez and Pirola [16] showed in passing that if $X \subseteq \mathbf{P}^3$ is a surface of degree $d \geq 4$, then $\text{cov. gon}(X) = d - 2$. Along similar lines, Fakhruddin established in his note [8] that given any integer $c > 0$, a very general hypersurface of sufficiently large degree in any smooth variety does not contain any curves of gonality $\leq c$. However as a measure of irrationality, it seems that the covering gonality was first studied systematically in [6], where Bastianelli computes $\text{cov. gon}(X)$ and bounds $\text{irr}(X)$ when $X = C_2$ is the symmetric square of a curve C .

The present work was most directly motivated by the interesting paper [4] in which Bastianelli, Cortini and De Poi consider the question of computing the degree of irrationality of a smooth projective hypersurface

$$X = X_d \subset \mathbf{P}^{n+1}$$

²We introduce the connecting gonality only because it fits naturally into the picture. In fact this invariant does not enter seriously into any of our results.

of degree d and dimension $n \geq 2$, generalizing the result of Noether for plane curves cited above. They show to begin with that if $d \geq n + 3$ then

$$(2) \quad d - n \leq \text{irr}(X) \leq d - 1.$$

It can happen that $\text{irr}(X) < d - 1$, but Bastianelli, Cortini and De Poi prove that if X is a *very general* surface of degree $d \geq 5$ or threefold of degree $d \geq 7$ then

$$\text{irr}(X) = d - 1,$$

and they classify the exceptional cases in these dimensions. They conjectured that the same inequality holds for a very general hypersurface $X \subseteq \mathbf{P}^{n+1}$ of arbitrary dimension $n \geq 2$ and degree $d \geq 2n + 1$, and that moreover if $d \geq 2n + 2$ then any dominant rational mapping

$$f : X \dashrightarrow \mathbf{P}^n \text{ with } \deg(f) = d - 1$$

is given by projection from a point of X .

Our first results concern covering gonality.

Theorem A. *Let $X \subseteq \mathbf{P}^{n+1}$ a smooth hypersurface of dimension n and degree $d \geq n + 2$. Then*

$$\text{cov. gon}(X) \geq d - n.$$

Observe that one recovers in particular the lower bound (2) of Bastianelli, Cortini and De Poi on the degree of irrationality of such hypersurfaces. In fact it suffices in the Theorem that X is normal with at worst canonical singularities, and in this setting the statement is best possible for every $n \geq 2$ and $d \geq n + 2$. More generally, we prove that the conclusion of the Theorem holds for any smooth projective variety X with

$$K_X \equiv_{\text{lin}} B + E$$

where B is a $(d - n - 2)$ -very ample divisor on X and E is effective.³ Thus we deduce

Corollary B. *Let M be a smooth projective variety, and let A be a very ample divisor on M . There is an integer $e = e(M, A)$ depending only on M and A with the property that if*

$$X_d \in |dA|$$

is any smooth divisor, then

$$\text{cov. gon}(X_d) \geq d - e.$$

In particular, the degree of irrationality of X_d goes to infinity with d . (One can prove this last fact directly using the ideas of [16], [6], [4], and [9]: see Remark 2.2.) As noted above, Fakhruddin proved in [8] the closely related result that in the situation of the Corollary, there is a linear function $d(c)$ such that a very general divisor $X_d \in |dA|$ actually contains no curves of gonality $\leq c$ provided that $d \geq d(c)$. (Compare Proposition 2.6 below.)

³Recall that a divisor B on a smooth projective variety Y is said to be p -very ample if any finite subscheme $\xi \subseteq Y$ of length $(p + 1)$ imposes independent conditions on $H^0(Y, B)$. If A is a very ample divisor then $\mathcal{O}_Y(pA)$ is p -very ample, and therefore if $X \subseteq \mathbf{P}^{n+1}$ is a smooth hypersurface of degree d then K_X is $(d - n - 2)$ -very ample.

Returning to smooth hypersurfaces in projective space, our second theorem proves the conjecture of Bastianelli, Cortini and De Poi under a slightly more stringent degree hypothesis.

Theorem C. *Let $X \subseteq \mathbf{P}^{n+1}$ be a very general smooth hypersurface of dimension n and degree $d \geq 3n$. Then*

$$\mathrm{irr}(X) = d - 1.$$

Furthermore, any rational mapping

$$f : X \dashrightarrow \mathbf{P}^n \quad \text{with } \deg(f) = d - 1$$

is given by projection from a point of X .

In their appendix [5] to the present paper, Bastianelli and De Poi prove that one can weaken the hypothesis to $d \geq 3n - 2$.

The proof of Theorem A, which is quite quick and elementary, occupies §1: there we work on an arbitrary smooth variety whose canonical bundle satisfies a suitable positivity property. For Theorem C, which appears in §2, we start with the set-up established by Bastianelli, Cortini and De Poi in [4]. Those authors give a very nice argument to show that if

$$f : X \dashrightarrow \mathbf{P}^n$$

is a rational mapping with

$$d - n \leq \deg(f) \leq d - 2,$$

then each of the fibres of f spans a line in \mathbf{P}^{n+1} provided that $d \geq 2n + 1$. They prove that these lines form a congruence of order one on \mathbf{P}^{n+1} , meaning that a general point of \mathbf{P}^{n+1} lies on exactly one of the lines. The main effort in [4] was to use results on the classification of such congruences to show that X must contain a rational curve when $n = 2$ or $n = 3$, which forces X to be special provided that $d \geq 2n + 1$. Our observation is that in arbitrary dimension n , whether or not X contains a rational curve one can locate on X a curve of gonality $\leq n$. On the other hand, drawing on computations of the first author and Voisin in [7], [20] one easily proves that a very general hypersurface of degree $d \geq 3n$ does not contain a curve of gonality $\leq n$, and Theorem C follows. The common thread in these arguments is that the invariants we consider are ultimately controlled by measuring the positivity of canonical bundles.

In §3 we discuss a few open problems. For example, it is natural to ask for extensions of the present results to other classes of varieties. More ambitiously, we feel that it would be very interesting to know whether any of the new techniques introduced in [13], [21] and [19] to study questions of rationality have any applicability here, or whether geometric or arithmetic connections such as those established in [1], [12], [15] for the gonality of curves extend to a higher-dimensional setting.

The reader will see that the methods of the present paper are very elementary, and several of the ideas involved are at least implicit in earlier work such as [16], [8], [6], [4] and [9]. However we have tried to pull things together in a natural way by focusing on a specific birational measure of positivity for the canonical bundle (Definition 1.1). We hope

that this might help to lay the foundation for further work on what we consider to be an interesting circle of questions.

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We are honored to dedicate this paper to János Kollár on the upcoming occasion of his sixtieth birthday. Beyond guiding the direction of algebraic geometry over three decades, János has been instrumental to the work of the first two authors through his encouragement and generosity with ideas. It is a pleasure to have this opportunity to express our admiration and thanks.

Concerning notation and conventions – we work throughout over the complex numbers. As usual, morphisms are indicated by solid arrows, while rational mappings are dashed. We have taken the customary liberties in confounding line bundles and divisors.

1. COVERING GONALITY

In this section we study the covering gonality of a projective variety X , and prove Theorem A from the Introduction. The basic strategy is to bound $\text{cov.gon}(X)$ in terms of the positivity of the canonical bundle K_X . So we start with some remarks on birational measures of positivity for line bundles.

Let X be an irreducible projective variety. Given an integer $p \geq 0$, recall that a line bundle L on X is said to be p -very ample if the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_\xi)$$

is surjective for every finite subscheme $\xi \subseteq X$ of length $p + 1$. In other words, one asks that every subscheme of length $p + 1$ imposes independent conditions on the sections of L . The condition we focus on here is a birational analogue of this.

Definition 1.1. A line bundle L on X *satisfies property* $(\text{BVA})_p$ if there exists a proper Zariski-closed subset $Z = Z(L) \subsetneq X$ depending on L such that

$$(1.1) \quad H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_\xi)$$

surjects for every finite subscheme $\xi \subset X$ of length $p + 1$ whose support is disjoint from Z .

Thus $(\text{BVA})_0$ is equivalent to requiring that L be effective, and $(\text{BVA})_1$ is what is often called “birationally very ample.”⁴

The following remarks yield a supply of examples.

Example 1.2. Let X be an irreducible projective variety.

⁴Hence “BVA.”

- (i). If L is a line bundle on X satisfying $(\text{BVA})_p$ and E is an effective divisor on X , then $\mathcal{O}_X(L + E)$ satisfies $(\text{BVA})_p$.
- (ii). Suppose that $f : X \rightarrow Y$ is a birational morphism of irreducible projective varieties. If L is a line bundle on Y satisfying $(\text{BVA})_p$, then f^*L satisfies $(\text{BVA})_p$ on X .
- (iii). More generally, let $f : X \rightarrow Y$ be a morphism which is birational onto its image, and suppose that L satisfies $(\text{BVA})_p$ on Y . Assume moreover that $f(X)$ is not contained in the exceptional set $Z \subseteq Y$ arising in the definition of property (BVA) . Then f^*L satisfies $(\text{BVA})_p$ on X .
- (iv). Suppose that

$$f : X \rightarrow \mathbf{P}$$

is a morphism from X to some projective space which is birational onto its image. Then $f^*\mathcal{O}_{\mathbf{P}}(p)$ satisfies $(\text{BVA})_p$.

- (v). Suppose that

$$X \subseteq \mathbf{P}^{n+1}$$

is a normal hypersurface of degree $d \geq n + 2$ with at worst canonical singularities, and let $\mu : X' \rightarrow X$ be a resolution of singularities. Then the canonical bundle $K_{X'}$ of X' satisfies $(\text{BVA})_{d-n-2}$.

Indeed, (i), (ii) and (iii) are clear from the definition, while (iv) is a consequence of (ii) and the elementary fact that $\mathcal{O}_{\mathbf{P}}(p)$ is p -very ample. For (v), it follows from the definition of canonical singularities that

$$K_{X'} \equiv_{\text{lin}} (d - n - 2)H + E,$$

where H is the pullback of the hyperplane bundle on X and E is effective. So the assertion follows from (i) and (iv). \square

The relevance of this notion to questions of gonality arises from the following elementary observation.

Lemma 1.3. *Let C be a smooth projective curve of genus g whose canonical bundle K_C satisfies $(\text{BVA})_p$. Then*

$$\text{gon}(C) \geq p + 2.$$

Proof. We may suppose $g \geq 2$. Let A be a globally generated line bundle of degree $d \leq g - 1$ on C . Then the divisor ξ of any section of A fails to impose independent conditions on $|K_C|$. Hence if K_C satisfies $(\text{BVA})_p$ then one must have $d \geq p + 2$. \square

We now turn to coverings by curves of specified gonality. Let X be an irreducible projective variety.

Definition 1.4. A *covering family of curves of gonality c* on X consists of a smooth family

$$\pi : \mathcal{C} \rightarrow T$$

of irreducible projective curves parametrized by an irreducible variety T , together with a dominant morphism

$$f : \mathcal{C} \longrightarrow X,$$

satisfying:

- (i). For a general point $t \in T$, the fibre $C_t =_{\text{def}} \pi^{-1}(t)$ is a smooth curve with $\text{gon}(C_t) = c$; and
- (ii). For general $t \in T$, the map $f_t : C_t \longrightarrow X$ is birational onto its image.

By standard arguments, the existence of such a family is equivalent to asking that X contains a (possibly singular) curve of gonality c passing through a general point.

Remark 1.5. We make some remarks about the formal properties of this definition.

- (i). After replacing T by a desingularization, one can suppose without loss of generality that T and \mathcal{C} are non-singular.
- (ii). Given a covering family as above, after restricting to a suitable subvariety of T we may suppose without loss of generality that $\dim \mathcal{C} = \dim X$, so that in particular the morphism

$$f : \mathcal{C} \longrightarrow X$$

is generically finite.

- (iii). Suppose that $\pi : \mathcal{C} \longrightarrow T$, $f : \mathcal{C} \longrightarrow X$ is a covering family with \mathcal{C} and T non-singular, and let $\nu : \mathcal{C}' \longrightarrow \mathcal{C}$ be the blowing up of \mathcal{C} along a smooth center. Then there is a non-empty Zariski-open subset $T_0 \subseteq T$ over which the restrictions of the two maps

$$\mathcal{C}' \longrightarrow T \quad , \quad \mathcal{C} \longrightarrow T$$

coincide. (Since blowing up along a divisor has no effect, we can assume that this center has codimension ≥ 2 , and hence maps to a subset of T having codimension ≥ 1 .)

- (iv). Let $\pi : \mathcal{C} \longrightarrow T$, $f : \mathcal{C} \longrightarrow X$ be a covering family with \mathcal{C} and T smooth, and let

$$\mu : X' \longrightarrow X$$

be a birational morphism. Then there is a non-empty Zariski-open subset $T_0 \subseteq T$ so that the restriction $\pi_0 : \mathcal{C}_0 \longrightarrow T_0$ extends to a family

$$f' : \mathcal{C}_0 \longrightarrow X'.$$

(In fact, by a suitable sequence of blow-ups, we can construct a modification $\mathcal{C}' \longrightarrow \mathcal{C}$ that admits an extension $f' : \mathcal{C}' \longrightarrow X'$. The assertion then follows from (iii).)

As in the Introduction, we focus on the smallest gonality of such a covering family:

Definition 1.6. The *covering gonality* $\text{cov. gon}(X)$ of X is the least integer $c > 0$ for which such a covering family exists.

It follows from Remark 1.5 (iv) that this is indeed a birational invariant.

Example 1.7. (Examples of covering gonality). Here are some examples where the covering gonality can be estimated or computed.

(i). Let X be a $K3$ surface. By a theorem of Bogomolov and Mumford ([17, p. 351]) X is covered by (singular) elliptic curves. Hence $\text{cov.gon}(X) = 2$.

(ii). Let $X = C_2$ be the symmetric square of a smooth curve C of genus g . Then X is covered by copies of C via the double covering $C \times C \rightarrow X$. Bastianelli [6] shows that these curves compute the covering gonality of X , ie $\text{cov.gon}(X) = \text{gon}(C)$.

(iii). Let $X \subseteq \mathbf{P}^3$ be a smooth surface of degree $d \geq 4$, let $x \in X$ be a general point, and let $T_x \subseteq \mathbf{P}^3$ be the tangent plane to X at x . Then

$$D_x = T_x \cap X$$

is an irreducible plane curve of degree d with a double point, which has gonality $d - 2$. Therefore $\text{cov.gon}(X) \leq d - 2$. In fact, Lopez and Pirola [16] show that this is the unique family of minimal gonality, and hence $\text{cov.gon}(X) = d - 2$.

(iv). Suppose now that $X \subseteq \mathbf{P}^4$ is a smooth threefold of degree $d \geq 5$. A dimension count predicts that X should be covered by a two-dimensional family of plane curves of degree d with triple points. One can prove – either directly or (as Jason Starr pointed out) by a degeneration – that this is indeed the case. Hence

$$\text{cov.gon}(X) \leq d - 3,$$

and the same inequality holds *a fortiori* for hypersurfaces of degree d and larger dimension. In general, one expects the covering gonality of hypersurfaces of given degree d to decrease as their dimension grows, until eventually they become rationally connected in the Fano range.

(v). Let $X \subseteq \mathbf{P}^{n+1}$ be a hypersurface of degree $d > n$ having an ordinary singular point $p \in X$ of multiplicity n : in particular, X has only canonical singularities. Projection from p gives rise to a rational map $X \dashrightarrow \mathbf{P}^n$ of degree $d - n$, and the inverse images of lines $\ell \subseteq \mathbf{P}^n$ then yield a covering of X by curves of gonality $\leq d - n$. Therefore $\text{cov.gon}(X) \leq d - n$, and it follows from Corollary 1.9 below that in fact $\text{cov.gon}(X) = d - n$. \square

The main theorem of this section asserts that the covering gonality of a smooth projective variety is bounded by the positivity of its canonical bundle.

Theorem 1.8. *Let X be a smooth projective variety, and suppose that there is an integer $p \geq 0$ such that its canonical bundle K_X satisfies property $(\text{BVA})_p$. Then*

$$\text{cov.gon}(X) \geq p + 2.$$

Proof. This is very elementary. Suppose that

$$\pi : \mathcal{C} \longrightarrow T, \quad f : \mathcal{C} \longrightarrow X$$

is a covering family of curves of gonality c . Thanks to Remark 1.5 (i) and (ii), there is no loss of generality in assuming that \mathcal{C} and T are smooth, and that f is generically finite.

Then

$$(*) \quad K_C \equiv_{\text{lin}} f^* K_X + E$$

where $E = \text{Ram}(f)$ is the ramification divisor of f . On the other hand, since π is smooth one has

$$(**) \quad K_{C_t} \equiv_{\text{lin}} K_C | C_t$$

for every $t \in T$. Furthermore, if $t \in T$ is general, then C_t meets the effective divisor E properly, and its image

$$f_t(C_t) \subseteq X$$

will not be contained in the exceptional set $Z(K_X) \subseteq X$ arising in Definition 1.1. Since by definition $f_t : C_t \rightarrow X$ is birational onto its image, it follows from (*), (**) and Remark 1.2 (iii) that K_{C_t} satisfies property (BVA) $_p$. Hence $c \geq p + 2$ thanks to Lemma 1.3. \square

Corollary 1.9. *Let $X \subseteq \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq n + 2$. Then*

$$\text{cov. gon}(X) \geq d - n.$$

The same statement holds if X is normal with only canonical singularities.

Note that if we allow canonical singularities, then Example 1.7 (v) shows that the statement is best possible for all $n \geq 2$ and $d \geq n + 2$. When $n = 1$ we recover Noether's result that a smooth plane curve of degree d has gonality $d - 1$.

Proof of Corollary. When X is smooth, its canonical bundle $\omega_X = \mathcal{O}_X(d - n - 2)$ is already $(d - n - 2)$ -very ample. For the second statement, we can pass to a desingularization, and then Example 1.2 (v) applies. \square

We observe next that a sufficiently positive divisor on any smooth variety has large covering gonality.

Corollary 1.10. *Let M be a smooth projective variety, and let A be a very ample line bundle on M . Fix an integer e such that $|(e + 2)A + K_M|$ is basepoint-free, and let*

$$X = X_d \in |dA|$$

be any smooth divisor. Then

$$\text{cov. gon}(X) \geq d - e.$$

Proof. In fact,

$$K_X = (K_M + dA) | X = ((d - e - 2)A + E) | X,$$

where $|E|$ is free. Since A is very ample, $\mathcal{O}_X((d - e - 2)A)$ is $(d - e - 2)$ -very ample, and therefore K_X satisfies Property (BVA) $_{d-e-2}$. \square

Finally, we say a word about the connecting gonality of an irreducible projective variety X . An evident modification of Definition 1.7 leads to the notion of a family of curves of gonality c connecting two general points of X , and as in the Introduction the least such gonality is defined to be $\text{conn.gon}(X)$. Clearly

$$\text{cov.gon}(X) \leq \text{conn.gon}(X),$$

and the example of a uniruled variety which is not rationally connected shows that the inequality can be strict. Unfortunately, we do not at the moment know any useful ways of controlling this invariant. For example, when X is the symmetric square of a curve of large genus, as in Example 1.7 (ii), we suspect that $\text{cov.gon}(X) < \text{conn.gon}(X)$, but we do not know how to prove this.

2. DEGREE OF IRRATIONALITY OF PROJECTIVE HYPERSURFACES

In this section we discuss the degree of irrationality and give the proof of Theorem C from the Introduction.

We start with some general remarks about the irrationality degree $\text{irr}(X)$ of an irreducible complex projective variety X of dimension n . Recall from the Introduction that this is defined to be the least degree of a dominant rational map

$$f : X \dashrightarrow \mathbf{P}^n.$$

Equivalently, one can characterize $\text{irr}(X)$ as the minimal degree of a field extension

$$\mathbf{C}(t_1, \dots, t_n) \subseteq \mathbf{C}(X)$$

where the $t_i \in \mathbf{C}(X)$ are algebraically independent rational functions on X . We refer to [22], [18], [23], [24], [6] for some computations and estimations of $\text{irr}(X)$, especially in the case of surfaces.

Given a rational covering $f : X \dashrightarrow \mathbf{P}^n$, observe that the inverse images of lines $\ell \subseteq \mathbf{P}^n$ determine a family of curves of gonality $\leq \deg(f)$ connecting two general points on X . This shows that

$$(2.1) \quad \text{cov.gon}(X) \leq \text{conn.gon}(X) \leq \text{irr}(X).$$

The existence of rationally connected varieties that are not rational – as well as many other examples – illustrates that the gonality invariants can be strictly smaller than $\text{irr}(X)$. However by combining (2.1) with Theorem 1.8 we find:

Corollary 2.1. *Let X be a smooth projective variety whose canonical bundle K_X satisfies Property (BVA) $_p$ for some $p \geq 0$. Then*

$$\text{irr}(X) \geq p + 2. \quad \square$$

As above (Examples 1.2 (v) and 1.7 (v)), equality holds for the desingularization of a hypersurface of degree d in \mathbf{P}^{n+1} with an ordinary n -fold point.

Remark 2.2. One can give a direct proof of (a strengthening of) the Corollary using results and methods of [16], [6], [4] and [9], involving correspondences with null trace and the Cayley-Bacharach property. Specifically, consider a dominant rational map

$$f : X \dashrightarrow Y$$

between two smooth projective n -folds. We claim:

$$(2.2) \quad \text{If } K_X \text{ satisfies } (\text{BVA})_p \text{ and } H^0(Y, K_Y) = 0, \text{ then } \deg f \geq p + 2.$$

In fact, given any rational covering f one has a trace map

$$\text{Tr}_f : H^0(X, K_X) \longrightarrow H^0(Y, K_Y)$$

on canonical forms. For $\eta \in H^0(X, K_X)$ and a general point $y \in Y$, one can view the value of $\text{Tr}_f(\eta)$ at y as being computed by averaging the values of η over the fibre $f^{-1}(y)$ of y . It follows that if $H^0(Y, K_Y) = 0$, then $f^{-1}(y)$ satisfies the Cayley-Bacharach property with respect to $|K_X|$: any n -form vanishing on all but one of the points of $f^{-1}(y)$ must vanish on the remaining one. (See for instance [4, Proposition 2.3] or [9, §3.2 – §3.4].) In particular, these points do not impose independent conditions on $H^0(X, K_X)$, and (2.2) follows. \square

Remark 2.3. Voisin has pointed out to us that one can also prove a variant of the statement (2.2) from the previous remark. Specifically, consider a smooth projective n -fold X with the property that the Hodge-structure $H^n(X, \mathbf{Q})_{\text{prim}}$ is irreducible: this holds for instance for a very general hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $> n + 2$. Suppose moreover that K_X satisfies property $(\text{BVA})_p$ with $p \geq 1$. If Y is *any* smooth projective variety of dimension n , then any rational covering

$$f : X \dashrightarrow Y \quad \text{with } \deg(f) < p + 2$$

must actually be birational. In fact, assume to the contrary that f is not birational. The mapping $f^*H^0(Y, K_Y) \longrightarrow H^0(X, K_X)$, which in any event is injective, must be surjective or zero, else it would give a non-trivial Hodge substructure of $H^n(X, \mathbf{Q})_{\text{prim}}$. The former possibility is impossible since K_X satisfies $(\text{BVA})_1$, and therefore it must be the case that $H^0(Y, K_Y) = 0$. Then the previous remark applies. (Compare [9, Proposition 3.5.2].)

We now turn to the case of a smooth hypersurface

$$X \subseteq \mathbf{P}^{n+1}$$

of dimension $n \geq 2$ and degree $d \geq n + 2$. Projection from a point of X shows that in any event

$$(2.3) \quad \text{irr}(X) \leq d - 1,$$

and by [4, Theorem 1.2] (or Corollary 1.9 above) one has the lower bound

$$(2.4) \quad \text{irr}(X) \geq d - n.$$

Example 2.4. Interestingly enough, it can actually happen that $\text{irr}(X) < d - 1$. For instance, suppose that $X \subseteq \mathbf{P}^3$ is a surface containing two disjoint lines $\ell_1, \ell_2 \subseteq X$. Then the line joining general points $p_1 \in \ell_1, p_2 \in \ell_2$ meets X at $(d - 2)$ residual points, and this defines a rational mapping

$$X \dashrightarrow \ell_1 \times \ell_2 \approx \mathbf{P}^2$$

of degree $d-2$. There are a few other examples of a similar flavor, and it is established in [4, Theorem 1.3] that these are the only surfaces of degree $d \geq 5$ in \mathbf{P}^3 for which $\text{irr}(X) = d-2$. In a similar way, if $X \subseteq \mathbf{P}^{2k+1}$ contains two disjoint k -planes then $\text{irr}(X) \leq d-2$, but apparently no examples are known of hypersurfaces of odd dimension ≥ 5 for which equality fails in (2.3). (See [4, 4.13, 4.14].) \square

Our goal in the rest of this section is to prove two results, which together will establish Theorem C from the Introduction.

Proposition 2.5. *Let $X \subseteq \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2n+1$, and suppose that*

$$f : X \dashrightarrow \mathbf{P}^n$$

is a rational covering of degree $\delta \leq d-1$. If f is not given by projection from a point of X , then there exists a (possibly singular) irreducible curve

$$C \subseteq X \text{ with } \text{gon}(C) \leq d - \delta.$$

Proposition 2.6. *Let $X \subseteq \mathbf{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. Then any irreducible curve $C \subseteq X$ satisfies*

$$\text{gon}(C) \geq d - 2n + 1.$$

Observe that Theorem C follows immediately from these two statements. In fact, suppose that $f : X \dashrightarrow \mathbf{P}^n$ is a covering of degree $\leq d-1$. Thanks to the lower bound (2.4), Proposition 2.5 guarantees that X contains a curve of gonality $\leq n$ unless f is projection from a point. On the other hand, Proposition 2.6 shows a very general hypersurface of degree $d \geq 3n$ contains no such curve.⁵

In preparation for the proof of Proposition 2.5, we start by summarizing some of the constructions and results of [4], upon which we will draw. The rational mapping $f : X \dashrightarrow \mathbf{P}^n$ is given by a correspondence

$$Z \subseteq X \times \mathbf{P}^n,$$

and for any $y \in \mathbf{P}^n$ we can view the fibre Z_y – which in general consists of δ distinct points of X – as a subset of \mathbf{P}^{n+1} . Bastianelli, Cortini and De Poi prove two key facts:

- (i). For general $y \in \mathbf{P}^n$ the fibre $Z_y \subseteq \mathbf{P}^{n+1}$ lies on a line

$$\ell_y \subseteq \mathbf{P}^{n+1}.$$

- (ii). A general point of \mathbf{P}^{n+1} lies on exactly one of these lines.

These are established in [4, Theorem 2.5, Lemma 4.1] using the ideas involving correspondences with null trace recalled in Remark 2.2 above; (i) is where the assumption $d \geq 2n+1$ comes into the picture. In classical language, the $\{\ell_y\}$ form a *congruence* of lines, ie a family of lines parametrized by an irreducible n -dimensional subvariety

$$B_0 \subseteq \mathbf{G} = \mathbf{G}(\mathbf{P}^1, \mathbf{P}^{n+1})$$

⁵In their appendix [5] to the present paper, Bastianelli and De Poi show that in the situation of Proposition 2.5, X actually contains a curve of gonality $\leq 3n-2$.

of the Grassmannian of lines in \mathbf{P}^{n+1} . Statement (ii) asserts that the congruence has *order one*: if $W_0 \subseteq B_0 \times \mathbf{P}^{n+1}$ is the restriction to B_0 of the tautological point-line correspondence in $\mathbf{G} \times \mathbf{P}^{n+1}$, this means that the projection

$$\mu_0 : W_0 \longrightarrow \mathbf{P}^{n+1}$$

is birational, and it implies that B_0 is rational.⁶

Replacing B_0 by a desingularization $B \longrightarrow B_0$, we arrive at the basic diagram:

$$(2.5) \quad \begin{array}{ccc} X & \xleftarrow{\mu'} & X' \\ \cap | & & \cap | \\ \mathbf{P}^{n+1} & \xleftarrow{\mu} & W \\ & & \downarrow \pi \text{ } \mathbf{P}^1\text{-bundle} \\ & & B \longrightarrow \mathbf{G} \end{array}$$

Here B is a smooth rational n -fold mapping birationally to its image in the Grassmannian \mathbf{G} , and $\pi : W \longrightarrow B$ is the pull-back to B of the tautological \mathbf{P}^1 -bundle on \mathbf{G} . The mapping $\mu : W \longrightarrow \mathbf{P}^{n+1}$ is birational, and we define $X' \subseteq W$ to be the proper transform of X in W . Thus X' is a reduced and irreducible divisor in W of relative degree δ over B , and $X' \longrightarrow B$ is a generically finite morphism of degree δ that represents birationally the original mapping $f : X \dashrightarrow \mathbf{P}^n$.

We now give the:

Proof of Proposition 2.5. Let

$$X^* = \mu^*(X) \subseteq W$$

be the full pre-image of X in W , so that X^* is a (possibly non-reduced) divisor in W of relative degree d over B . We can write $X^* = X' + F$, where F is a divisor of relative degree $d - \delta \geq 1$ over B . Now fix any irreducible component $Y \subseteq F$ that dominates B , and view Y as a reduced irreducible variety of dimension n . Thus Y sits in a diagram

$$\begin{array}{ccc} X & \longleftarrow & Y \\ \cap | & & \cap | \\ \mathbf{P}^{n+1} & \xleftarrow{\mu} W \xrightarrow{\pi} & B, \end{array}$$

and we have

$$0 < \deg(Y \longrightarrow B) \leq d - \delta.$$

Suppose first that $\mu(Y)$ consists of a single point $p \in X$. This means that all the lines in the congruence pass through p , and hence f must be projection from p . Therefore we

⁶If one fixes a general hyperplane $H \subseteq \mathbf{P}^{n+1}$, then almost every point of H lies on a unique line of the congruence, establishing a birational isomorphism $H \approx B_0$.

may assume that

$$(*) \quad \dim \mu(Y) \geq 1.$$

Now since B is rational, we can choose a rational curve $\Gamma \subseteq B$ joining two general points of B , and by Bertini we may as well suppose moreover that

$$D = Y \times_B \Gamma$$

is irreducible. Viewing D as a reduced curve, one has

$$\deg(D \rightarrow \Gamma) \leq d - \delta,$$

so $\text{gon}(D) \leq d - \delta$. Moreover thanks to $(*)$ we can suppose that

$$C =_{\text{def}} \mu(D) \subseteq X$$

also has dimension 1. It is conceivable that $D \rightarrow C$ is not birational, but it is elementary and well-known that gonality does not increase in coverings of curves. Therefore $\text{gon}(C) \leq d - \delta$, as required. \square

Remark 2.7. (Fundamental locus of a correspondence of order one). Consider an arbitrary order-one congruence Γ of lines in \mathbf{P}^{n+1} , given by a diagram as in (2.5)

$$\begin{array}{ccc} & W & \\ \mu \swarrow & & \searrow \pi \\ \mathbf{P}^{n+1} & & B. \end{array}$$

As explained for example in [2], a computation of canonical bundles shows that the exceptional divisor $E \subseteq W$ of μ has relative degree n over B ; its image

$$\mu(E) = \text{Fund}(\Gamma) \subseteq \mathbf{P}^{n+1}$$

is called the fundamental locus of Γ , and it consists of those points of \mathbf{P}^{n+1} through which infinitely many lines of the congruence pass. Assume that Γ is not the star of lines through a fixed point of \mathbf{P}^{n+1} . Then the fundamental locus has dimension ≥ 1 , and arguing as above one sees that $\text{Fund}(\Gamma)$ contains a curve of gonality $\leq n$. In their appendix [5] to the present paper, Bastiabeli and De Poi considerably strengthen this observation, while in [4] Bastianelli, Cantini and De Poi raise the interesting question whether in fact the fundamental locus always contains a rational curve. \square

Finally, we show that Proposition 2.6 follows immediately from computations of [7] and [20]. This is essentially the same argument that appears in [8].

Proof of Proposition 2.6. Let $S = H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}}(d))$ be the vector space of all hypersurfaces of degree d , which we view as an affine variety. Let

$$\mathcal{X} \subseteq S \times \mathbf{P}^{n+1}$$

be the universal hypersurface of degree d . Denote by

$$pr_1 : \mathcal{X} \rightarrow S, \quad pr_2 : \mathcal{X} \rightarrow \mathbf{P}^{n+1}$$

the two projections, and write $s = \dim S$.

Suppose now that a very general hypersurface of degree d contains a curve of gonality c . Then by a standard argument there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \pi \downarrow & & \downarrow pr_1 \\ T & \xrightarrow{\rho} & S, \end{array}$$

where $\pi : \mathcal{C} \rightarrow T$ is a family of curves of gonality c , ρ is étale, and $f_t : C_t \rightarrow X_{\rho(t)}$ is birational onto its image. In this setting, Ein and Voisin prove that if $t \in T$ is a general point, then

$$\Omega_{\mathcal{C}}^{s+1} \otimes \left((pr_2 \circ f)^* \mathcal{O}_{\mathbf{P}^{n+1}}(2n+1-d) \right) \Big|_{C_t}$$

is generically generated by its global sections (cf [20, Theorem 1.4]), where $C_t = \pi^{-1}(t)$ is the fibre of π . This implies that

$$K_{C_t} \equiv_{\text{lin}} (d-2n-1)H_{C_t} + (\text{effective}),$$

where H_{C_t} is the pull-back of the hyperplane bundle from \mathbf{P}^{n+1} . Thus K_{C_t} satisfies property (BVA) $_{d-2n-1}$, and Lemma 1.3 applies to show that $c \geq d-2n+1$. \square

3. OPEN PROBLEMS

In this final brief section, we mention a few open questions concerning this circle of ideas.

To begin with, it would be interesting to compute – or at least estimate – the degree of irrationality for various natural classes of examples. This seems non-trivial already in the case of surfaces. For example, suppose that X_d is a very general polarized $K3$ surface of degree d , (ie X_d carries an ample line bundle L_d with $\int c_1(L_d)^2 = 2d-2$.) Is it true that

$$\lim_{d \rightarrow \infty} \text{irr}(X_d) = \infty?$$

Note that here $\text{cov.gon}(X_d)$ is independent of d (Example 1.7 (i)). One can ask a similar question for abelian surfaces with a polarization of type $(1, d)$. Yoshihara [24], [18] has some partial results in this direction, but the overall picture is far from clear. Bastianelli [6, Remark 6.7] proposes a rather clean conjecture about what one might expect when X is the symmetric product of a curve.

In a similar vein, one would like to know the birational positivity (in the sense of Definition 1.1) of the canonical bundles of various specific varieties. For instance, suppose that $X = C_k$ is the k^{th} symmetric product of a curve of genus g and gonality c . Is it true (at least if $k \ll g$) that K_X satisfies (BVA) $_{c-2}$? When $k = 2$ this would “explain” [6, Theorem 1.6], and in any event is very close to the arguments in that paper.

A very natural extension of Theorem C would be to compute – or at least bound realistically – the irrationality invariants for a very general complete intersection $X \subseteq \mathbf{P}^{n+e}$ of e hypersurfaces of given degrees. The results of the previous sections yield some

statements, but they are additive in the degrees of the defining equations, whereas one might expect bounds that are closer to multiplicative, as in [14, 4.12] for curves.

As noted at the end of §1, we unfortunately have very little to say about the connecting gonality $\text{conn.gon}(X)$ of an irreducible projective variety X . It would be interesting to develop techniques that would yield bounds on this invariant. While many examples suggest that it is quite common for $\text{cov.gon}(X) \ll \text{irr}(X)$, it is not clear at the moment to what extent $\text{cov.gon}(X)$ and $\text{conn.gon}(X)$ can diverge.

In their paper [11], Heinzer and Moh point out that it is interesting to consider not only $\text{irr}(X)$, but all the possible degrees of a rational mapping

$$f : X \dashrightarrow \mathbf{P}^n.$$

When $n = \dim X = 1$ they remark that this is an additive semigroup, but in higher dimensions not much seems to be known about what integers can occur. One can ask the analogous question for the gonality of covering or connecting families of curves.

In a more speculative direction, a number of new techniques have been introduced to study questions of rationality, such as Kollár's passage to characteristic $p > 0$ [13], the Chow-theoretic ideas used by Voisin [21], and the combination of these by Totaro [19]. It would be very interesting if ideas along these lines could be used to say something about measures of irrationality. Similarly, the papers [15] and [12] show that the gonality of a curve C influences various Riemannian and Kähler invariants of varieties associated to C . Are there any analogous statements in higher dimensions?

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